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Robust Colocated Control for Large Flexible Space Structures

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Closed-loop performance properties of a large flexible space structure with colocated actuators and sensors are examined. It is shown that when an output feedback gain matrix is chosen as symmetric negative-definite, a linear quadratic optimization (LQ) problem can be stated and solved in closed form via an inverse optimal control formulation. The known properties of this LQ problem guarantees the stability of the closed-loop system. It is further shown that the closed-loop system remains stable in the presence of certain parameter variations. These robustness properties include also the effect of truncated modes on stability. The understanding of the specific output feedback controller, offered through the inverse optimal control solution, lends itself to a design procedure that selects the output feedback gain matrix in an iterative manner.

I. Introduction

THE active control of large space structures (LSS) has become a topic of major concern and interest in recent years (see, for example, Refs. 1-3), because mission performance is greatly affected by the structural flexibility after deployment. This flexibility problem has moved to the forefront of design concerns due to the extremely large physical dimensions envisioned for future space structures.

The modeling process of these space structures is greatly affected by their size which results in a mathematical model of high dimension. To expedite numerical solutions, this model is reduced in size to yield the design model. An important problem to be resolved is the effect of control, derived from the reduced-order model, on the higher-order model; this problem is known as the "control spillover."[4]

Increasing the low natural damping of the flexible structure is one of the main requirements of an active control system. One of the simplest forms of implementing this requirement is through rate feedback controllers where the actuators and sensors are placed together (colocated). For a system with a single input it can be shown that this form of a controller will result in a stable closed-loop system.

In this paper we are concerned with finding the general properties of an output feedback gain matrix that will result in a closed-loop stable system where the actuators and sensors are colocated. The approach taken toward this problem is through an inverse optimal control formulation.[5-8] This, in turn, allows us to use the well-studied properties of the linear quadratic (LQ) regulator for the stability problem at hand.

In Sec. II we state the problems to be treated, Sec. III presents the inverse optimal control problems and its solution, Sec. IV describes some specific properties of this solution, and Sec. V considers its robustness properties. In Sec. VI we construct a design procedure based on the theoretical development of the previous sections, and Sec. VII presents a numerical example.

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II. Problem Statement

The generalized colocated control problem is concerned with finding a stabilizing output feedback control law where the actuators and sensors are placed together (colocated). Neglecting the low natural damping of the flexible system, the finite-order linear oscillatory system is described through

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (x \in R^{2n}, \quad u \in R^m) \tag{1}$$

where

$$A = \left[\begin{array}{c|c} 0 & I_n \\ \hline -A_0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{c} 0 \\ \hline B_0 \end{array}\right] \tag{2}$$

and A_0 is an $n \times n$ symmetric positive-definite matrix, B_0 is an $n \times m$ matrix of full rank $m (m \leq n)$, and I_n is the $n \times n$ identity matrix. The output vector y is obtained then through

$$y(t) = B^T x(t) \qquad (y \in R^m) \tag{3}$$

The state vector x of the system described in Eq. (1) is defined by

$$x \triangleq [q, \dot{q}] \tag{4}$$

where q and $\dot{q}$ are generalized displacements and velocities respectively. Using the output vector y in implementing an output feedback control law will result in a velocity (rate) feedback only due to the structure of B in Eq. (2). This is an important property of colocated controls for systems described as in Eq. (2); it will be exploited in deriving the analytical foundations for the design procedure.

Consider now the case where Eqs. (1) and (2) describe a single-input single-output (SISO) system. Then, it can be shown that the use of a colocated rate feedback control will result in an eigenvalue shift to the left in the complex plane (increased damping). The proof of this property is based on Jacobi's formula for eigenvalues perturbations and the modal structure of the system in Eqs. (1) and (3).[9] Can the stability properties of colocated rate feedback control be extended for multi-input multi-output (MIMO) systems?

Another problem associated with the model described in Eqs. (1-3) is that of large dimensionality. This dimension depends on the fineness of the grid used in the application of the finite-element method. Usually, however, the dimension is quite large and some order reduction is required to facilitate numerical solutions. This reduction in order will result in modes that are truncated from the final model used for design. The effect of the control derived from the reduced-order model on the truncated modes ("control spillover"[4]) is of extreme importance when evaluating the selected control law.

The specific problems to be addressed in this paper are the following:

1) What are the general properties of the stabilizing output feedback gain matrix? This control law is given by

$$u(t)=Ky(t) \qquad (u\in R^m) \tag{5}$$

2) To what extent, if any, are closed-loop stability properties affected by truncated modes?

3) How are closed-loop stability properties affected by parameter variations (or uncertainty) in system matrices?

Problem 1 will be answered through an inverse optimal control problem formulation to be presented next.

III. Inverse Optimal Control Problem

The first problem stated in Sec. II is concerned with the general properties of the stabilizing output feedback control law. Suppose we have such a stabilizing control law. The question we ask now is under what conditions is this control also optimal in the sense that

$$J=\int_0^\infty [x^T(t)Qx(t)+u^T(t)Ru(t)]\,dt \tag{6}$$

is minimized. This problem is known as the inverse optimal control problem, whereby one starts with a control and finds for what Q and R matrices Eq. (6) assumes a minimum.

The motivation for looking at this formulation stems from the fact that the solution properties of this LQ problem have been studied extensively (see, for example, Ref. 6). If the given control law is optimal for the problem in Eq. (6), then

$$u^*=Ky=-R^{-1}B^TPx \tag{7}$$

where P is the $2n\times 2n$ positive-definite solution of the algebraic matrix Riccati equation (ARE) given by

$$PA+A^TP-PBR^{-1}B^TP+Q=0 \tag{8}$$

The inverse optimal control problem is, therefore, the following. Given an admissible (i.e., stabilizing) rate feedback control law find for what, if any, weight matrices Q and R, the ARE in Eq. (8) is solved.

Let the $2n\times 2n$ symmetric P and Q matrices be partitioned as

$$P=\left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^T & P_{22}\end{array}\right] \qquad Q=\left[\begin{array}{c|c} Q_{11} & 0 \\ \hline 0 & Q_{22}\end{array}\right] \tag{9}$$

Then from the ARE in Eq. (8), we find

$$P_{11}-A_0P_{22}-P_{12}B_0R^{-1}B_0^TP_{22}=0 \tag{10}$$

$$P_{12}A_0+A_0P_{12}^T+P_{12}B_0R^{-1}B_0^TP_{12}=Q_{11} \tag{11}$$

$$P_{22}B_0R^{-1}B_0^TP_{22}-P_{12}P_{12}^T=Q_{22} \tag{12}$$

Since a rate feedback control law is used, we find from Eq. (7) that

$$B_0^TP_{12}=0 \tag{13}$$

Therefore, P_{12} can be expressed as

$$P_{12}=N_1\Phi N_1^T \tag{14}$$

where

$$B_0^TN_1=0 \tag{15}$$

$$\Phi=\Phi^T>0 \tag{16}$$

The $n\times n$ matrix P_{12} as given in Eq. (14) is symmetric and, from Eq. (16), positive-semidefinite. The latter condition is required to satisfy Eq. (11) since $(-A_0)$ is a stable matrix.

By observing Eq. (12) it is easy to verify that for a dynamic system under a rate feedback control law, the $n\times n$ matrix P_{12} in Eqs. (10-12) has to satisfy

$$P_{12}=0 \tag{17}$$

Since $P_{12}=0$, the resulting $2n\times 2n$ matrix P that solves the ARE will be a block diagonal matrix. This motivates the following form for the matrix P to be checked as a possible solution of the ARE. Let

$$P=\alpha B(B^TB)^{-1}B^T+NEN^T \qquad (\alpha>0) \tag{18}$$

where

$$B^TN=0 \qquad (N\in R^{2n\times(2n-m)}) \tag{19}$$

$$N^TN=I_{2n-m} \tag{20}$$

$$E=E^T>0 \qquad (E\in R^{(2n-m)\times(2n-m)}) \tag{21}$$

Is is clear that P in Eq. (18) is symmetric. To show that it is also positive-definite we note that

$$PB=\alpha B \tag{22}$$

$$PN=NE \tag{23}$$

Therefore, B and N are right eigenvectors of the matrix P corresponding to positive eigenvalues; therefore, P is positive-definite. Similar results are obtained for left eigenvectors.

Using the form for P as given in Eq. (18) the ARE can be written as

$$PA+A^TP-\alpha^2BR^{-1}B^T+Q=0 \tag{24}$$

Since

$$B=\left[\begin{array}{c} 0 \\ \hline B_0\end{array}\right]$$

we find that to satisfy Eq. (19)

$$N=\left[\begin{array}{c|c} 0 & I_n \\ \hline N_0 & 0\end{array}\right] \qquad (N_0\in R^{n\times(n-m)}) \tag{25}$$

where N_0 is chosen such that

$$B_0^TN_0=0 \tag{26}$$

$$N_0^TN_0=I_{n-m} \tag{27}$$

Let the symmetric matrix E in Eq. (18) be partitioned as follows

$$E=\begin{array}{c} \left[\begin{array}{c|c} E_1 & E_2 \\ \hline E_2^T & E_3\end{array}\right]\begin{array}{l}\}\,n-m \\ \}\,n\end{array} \\ \begin{array}{cc}\underbrace{\quad}_{n-m} & \underbrace{\quad}_{n}\end{array}\end{array} \tag{28}$$

It can be shown (see the Appendix for details) that a necessary and sufficient condition for P in Eq. (18) to solve the ARE in Eq. (8) is that

$$Q=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha^2 B_0 R^{-1} B_0^T \end{array}\right] \tag{29}$$

$$E=\left[\begin{array}{c|c} \alpha I_{n-m} & 0 \\ \hline 0 & \alpha A_0 \end{array}\right] \tag{30}$$

The $2n \times 2n$ positive-definite solution matrix P is given by

$$P=\left[\begin{array}{c|c} \alpha A_0 & 0 \\ \hline 0 & \alpha I_n \end{array}\right] \tag{31}$$

Once the solution to the ARE is obtained, one can find the optimal control as well. The case under consideration in this paper is that where the sensors and actuators are colocated. Therefore, it is easily verified that when the output feedback control law is chosen as

$$u=Ky \tag{32}$$

where K is a symmetric, $m \times n$ negative-definite matrix and the output is given through

$$y=B^T x \tag{33}$$

then

$$R=-\alpha K^{-1} \tag{34}$$

is the $m \times m$ control penalty matrix of the quadratic objective function

$$J=\int_0^\infty (x^T Q x + u^T R u)\,\mathrm{d}t \tag{35}$$

The results of this section are summarized in Table 1.

Table 1 Summary of Sec. III

Given:

$\dot{x}(t)=Ax(t)+Bu(t) \qquad (x \in R^{2n},\ u \in R^m)$

$u(t)=KB^T x(t)$ (rate feedback)

$K=K^T<0$ (a *given* gain matrix)

$$A=\left[\begin{array}{c|c} 0 & I_n \\ \hline -A_0 & 0 \end{array}\right] \qquad B=\left[\begin{array}{c} 0 \\ \hline B_0 \end{array}\right] \qquad A_0=A_0^T>0$$

$$x \triangleq [q,\dot{q}]^T \qquad x(0)=x_0$$

Problem:

For what Q and R matrices is u the optimal control for a linear quadratic objective function given by

$$J^*=\min_u \int_0^\infty (x^T(t) Q x(t) + u^T(t) R u(t))\,\mathrm{d}t$$

Solution of Ricatti equation:

$$P=\alpha\left[\begin{array}{c|c} A_0 & 0 \\ \hline 0 & I_n \end{array}\right]$$

Input penalty matrix:

$$R=-\alpha K^{-1} \qquad (\alpha>0)$$

State penalty matrix:

$$Q=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha^2 B_0 R^{-1} B_0^T \end{array}\right]$$

Optimal cost:

$$J^*=x_0^T P x_0=\alpha(q_0^T A_0 q_0+\|\dot{q}_0\|^2)$$

IV. Solution Properties

In Sec. II, we stated that the closed-loop system with colocated sensors and actuators under a scalar input is stable. The proof was based on Jacobi's formula for eigenvalue perturbations. The solution obtained in the previous section can be used to prove the closed-loop stability properties of the system under the control law derived in Eq. (32), which will extend the known colocated control properties to include the multivariable case.

The stabilization of a linear system by pole-placement via state feedback requires the system given by

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{36}$$

to be fully controllable.[10] (This requirement can be relaxed to that of stabilizability.[11]) This is an important structural property that will be assumed for the system under consideration.

It is well known[6] that if $Q=L^T L$, where (A,L) is an observable pair, then the ARE has only one positive-definite solution, P, and the optimal control yields an asymptotically stable closed-loop system. In the case discussed in the previous sections, we have

$$Q=L^T L \tag{37}$$

where the $m \times 2n$ matrix L is given by

$$L=[0,\alpha R^{-1/2} B_0^T] \tag{38}$$

It can be verified that if (A,B) is a controllable pair, R and A_0 are symmetric positive-definite matrices and $\alpha \neq 0$ then (A,L) is an observable pair.

Note: This observability property shows that a system under the control given by Eq. (32) will be asymptotically stable.

The results obtained thus far can be used to yield a physical interpretation of the solution to the inverse optimal control problem.

Since the state vector x is partitioned according to $x=[q,\dot{q}]^T$, the optimal performance is written as

$$J^*=\min_u [J_1+J_2] \tag{39}$$

where

$$J_1=\alpha^2\int_0^\infty \dot{q}^T B_0 R^{-1} B_0^T \dot{q}\,\mathrm{d}t \tag{40}$$

$$J_2=\int_0^\infty u^T R u\,\mathrm{d}t \tag{41}$$

The interpretation suggested above is that the cost functional is the total energy of the system. The term contributed by the state is the kinetic energy where

$$E=\dot{q}^T M \dot{q} \tag{42}$$

and the "mass" M is given by

$$M=\alpha^2 B_0 R^{-1} B_0^T \tag{43}$$

Also, the use of velocity feedback where the actuators and sensors are colocated results in an LQ problem where only kinetic energy is weighted. (The term "kinetic energy" is used because of the form of Eq. (40), a quadratic functional of velocity; this is not, however, the true kinetic energy of the structure).

V. Robustness Properties

As mentioned in Sec. II, one of the problems associated with modeling large flexible space structures is that of truncated modes. The matrix describing the dynamics of the flexible system, given by Eq. (3) has eigenvalues distributed on the imaginary axis. The process of mode truncation is carried out by ignoring eigenvalues that are far from the origin on the imaginary axis (high frequencies). Analytically, the problem of mode truncation is as follows. Let the full-order finite linear system† be described by

$$\dot z(t) \triangleq \begin{bmatrix} \dot x(t) \\ \dot x_T(t) \end{bmatrix} = \left[\begin{array}{cc|c} 0 & & I \\ \hline -A_0 & 0 & 0 \\ \hline 0 & -A_T & \end{array}\right] z(t) + \begin{bmatrix} 0 \\ B_0 \\ B_T \end{bmatrix} u(t) \tag{44}$$

where the subscript T stands for the part of the system to be truncated. The output used for control is given by

$$y(t)=B_0^T x(t)+B_T^T x_T(t) \qquad (y\in R^m) \tag{45}$$

The matrix A_T contributes to the full-order system frequencies higher than those contributed by A_0 and, therefore, can be truncated to yield the reduced order model described by

$$\dot x(t)=\left[\begin{array}{c|c} 0 & I_n \\ \hline -A_0 & 0 \end{array}\right] x(t)+\begin{bmatrix} 0 \\ B_0 \end{bmatrix} u(t) \tag{46}$$

The first robustness result can now be stated.

Theorem 1: The full-order system described in Eq. (44) under the control

$$u(t)=Ky(t) \qquad -K=-K^T>0 \tag{47}$$

and y given by Eq. (45), will remain stable for every A_T provided that (A_0,B_0), (A_T,B_T) are controllable pairs.

Proof: Using the control as derived from the reduced-order system results in the full-order system assuming the form given by

$$z(t)=\left[\begin{array}{cc|c} 0 & & K \\ \hline -A_0 & 0 & \left(\dfrac{B_0}{B_T}\right)K(B_0{:}B_T)^T \\ \hline 0 & -A_T & \end{array}\right] z(t)$$

†The real flexible system is described through a second order PDE (wave equation). In linear form, the flexible system is of infinite dimension. Equation (44), therefore, considers only a finite number of these modes.

The solution to the inverse optimal control problem and the controllability of the system indicate that this system will be asymptotically stable provided (A_0,B_0) and (A_T,B_T) form controllable pairs. ■

This result shows how the inverse optimal control solution can be put to use to answer the question of control spillover.[4]

Observing the derivation of the inverse optimal control solution we can arrive at the second robustness result. This result shows how stability properties of the closed-loop system are affected by parameter variations in the system matrix A_0.

Corollary: Consider the dynamic system described in Eq. (46) and where the $n\times n$ matrix A_0 can have uncertain parameters that belong to a certain set. Applying the output feedback control of Eq. (47) will result in a stable closed-loop system for every A_0 in the given range of possible parameter variation provided A_0 is always symmetric and positive-definite over its entire range. ■

This property holds true since the solution of the inverse optimal control problem requires A_0 to be positie-definite but its exact value is not required. As long as K is chosen as a symmetric negative-definite matrix stability properties are insured. The value of the cost functional will, of course, vary with different A_0.

Remarks: 1) in the case when A_0 is a positive-definite $n\times n$ matrix, the above result guarantees stability of the closed-loop system provided that A_0 remains positive-definite over all possible variations. 2) The stability problem of colocated controllers was also discussed in Refs. 12 and 13, although from a different approach.

VI. Design Procedure

It was shown above that the solution of the inverse optimal control problem results in a state weighting matrix given by

$$Q=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha^2 B_0 R^{-1} B_0^T \end{array}\right] \tag{48}$$

In practice, however, one starts the design procedure by specifying the weighting matrices rather than obtaining them as an end result. This specification results from the performance levels the system is required to achieve.

The use of low-authority controllers (colocated sensors and actuators) to date was concerned mainly with the question of stability. The results obtained above allow us to extend the scope of these controllers beyond that of stability.

Let the state weighting matrix Q, be specified as

$$Q=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q_0 \end{array}\right] \tag{49}$$

where Q_0 is any $n\times n$ symmetric positive-semidefinite matrix.

Given Q as in Eq. (49), the design is concerned with solving

$$-\alpha B_0 K B_0^T = Q_0 \tag{50}$$

for the symmetric gain matrix K.

If the $n\times n$ matrix B_0 is square and nonsingular, i.e., $n=m$, then the required gain is obtained from Eq. (50) as

$$K=-(1/\alpha)B_0^{-1}Q_0(B_0^T)^{-1} \tag{51}$$

In practice, however, one finds that $m<n$ in which case Eq. (51) will not be available. A valid assumption that can still be made is that B_0 is of full rank equal to m.

In this case we can solve Eq. (50), so that the solution is approximate in the least-squares sense. Let

$$J=\mathrm{tr}[(Q_0+\alpha B_0 K B_0^T)(Q_0+\alpha B_0 K B_0^T)^T] \tag{52}$$

then, setting

$$\left.\frac{\partial J}{\partial K}\right|_{K=K^*}=0 \qquad (53)$$

we find

$$K^*=-\left(\frac{1}{\alpha}\right)(B_0^TB_0)^{-1}B_0^TQ_0B_0(B_0^TB_0)^{-1} \qquad (54)$$

Remarks: 1) Since Q_0 is positive-definite, and $\alpha>0$, the gain matrix obtained in Eq. (54) is symmetric negative-definite as required. 2) Since the objective function of Eq. (52) is convex, the gain matrix of Eq. (54) is globally optimal.

Using the gain as given in Eq. (54) for the control of the system in Eq. (2) through the application of Eq. (6) will result in a solution of an inverse optimal control problem where the actual state weighting matrix is given by

$$\hat{Q}=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q^* \end{array}\right] \qquad (55)$$

and

$$Q^*=B_0(B_0^TB_0)^{-1}B_0^TQ_0B_0(B_0^TB_0)^{-1}B_0^T \qquad (56)$$

The actual weighting matrix Q^* of Eq. (56) can be rewritten as

$$Q^*=CQ_0C \qquad (57)$$

where

$$C=B_0(B_0^TB_0)^{-1}B_0^T=C^T \qquad (58)$$

The design procedure is concerned with specifying Q_0 so that the resulting Q^* as given in Eq. (57) will approach a desired weighting structure i.e.,

$$Q^*\approx Q_d \qquad (59)$$

The above discussion can be summarized into a numerical procedure.

The numerical procedure is motivated by Eq. (59). The matrix Q_0 is the design matrix to be chosen in an iterative manner until system's closed-loop performance is acceptable.

The procedure is as follows:

Step 1. Read $A_0, B_0, Q_0, m, n,$ and α.

Step 2. Set $k=1, Q_k=Q_0, D=B_0(B_0^TB_0)^{-1}$, and $C=DB_0^T$.

Step 3. $\quad Q^*=CQ_kC \qquad K_k^*=-(1/\alpha)D^TQ_kD$

Find eigenvalues of $\left[\begin{array}{c|c} 0 & I_n \\ \hline -A_0 & B_0K_kB_0^T \end{array}\right]$

Step 4. If eigenvalue distribution is acceptable (damping ratios, frequencies, etc.) go to step 5; otherwise find the diagonal difference of Q^* and Q_k:

$$\gamma_i=Q^*(i,i)-Q_k(i,i) \qquad (1\le i\le n)$$

then

$$Q_{k+1}(i,i)=Q_k(i,i)-\zeta_i\operatorname{sgn}(\gamma_i) \qquad (\zeta_i>0)$$

as $k\to k+1$. Go to step 3.

Step 5. STOP.

Remarks: 1) The choice of the parameter ζ_i can be varying, depending on present eigenvalue distribution. 2) The termination of this numerical procedure is dictated by engineering judgment rather than convergence properties.

This numerical procedure is used in the next section to design a controller for a flexible structure (a pyramid).

VII. Generalized Colocated Control Design

The flexible structure considered in this example was recently designed by K. Soosaar and R. Strunce of the Charles Stark Draper Laboratory (CSDL). This structure consists of a tetrahedral truss supported on the ground by three right-angled bipods (see Fig. 1). Elastic flexibility of truss members is assumed to be in an axial direction only. This structure can be described by a model having 12 structural modes due to the three degrees-of-freedom of each of the four vertices.

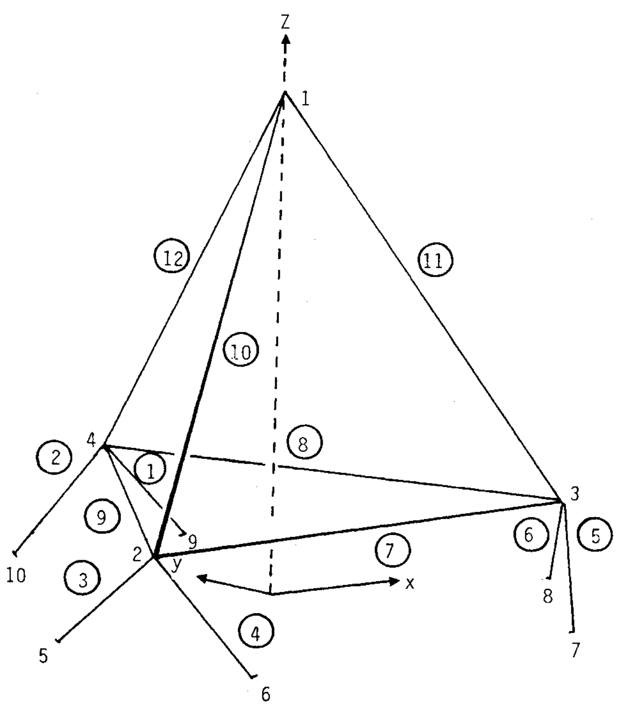

Fig. 1 The CSDL example structural model.

Table 2 Cross-sectional areas

Truss element no.	Area
1	100
2	100
3	100
4	100
5	100
6	100
7	1000
8	1000
9	1000
10	1000
11	100
12	100

Table 3 Structural node coordinates

Node	X	Y	Z
1	0.0	0.0	10.165
2	−5.0	−2.887	2.0
3	5.0	−2.887	2.0
4	0.0	5.7735	2.0
5	−6.0	−1.1547	0.0
6	−4.0	−4.6188	0.0
7	4.0	−4.6188	0.0
8	6.0	−1.1547	0.0
9	2.0	5.7735	0.0
10	−2.0	5.7735	0.0

Table 4 Design iterations

Open-loop frequency	Iteration 1		Iteration 2		Iteration 3		Iteration 4		Iteration 5	
	Weighting	ζ, %	Weighting	ζ, %	Weighting	ζ, %	Weighting	ζ, %	Weighting	ζ, %
1.34	10	0.55	100	0.97	100	1.12	200	1.66	400	2.56
1.66	10	2.1	100	4.43	100	4.4	100	3.99	200	5
2.89	0	0.62	0	1.4	0	1.73	0	2.47	0	3.4
2.96	10	7.4	10	29.7	10	36	5	29.2	4	60.4
3.39	10	25.8	4	4.63	10	8.15	8	8.02	10	5.8
4.21	0	7.72	0	5.23	0	10.87	0	10.3	0	13.8
4.56	0	0.61	0	22.5	0	25	0	29.97	0	61.6
4.75	0	3.84	0	0.83	0	0.83	0	0.98	0	1.3
8.54	0	6.26	0	16	0	17	0	16.04	0	26.7
9.25	0	2.82	0	1.07	0	1.16	0	1.18	0	0.79
10.3	0	1.84	0	0.93	0	1.96	0	1.64	0	2
12.9	0	0.33	0	0.22	0	0.34	0	0.31	0	0.33

The stiffness properties of the structure are expressed in terms of cross-sectional areas of truss members. These areas are given in Table 2. Node coordinates are given in Table 3. Analytically, the structure is described through

$$\dot{x} = \left[\begin{array}{c|c} 0 & I_n \\ \hline -A_0 & 0 \end{array}\right] x + \left[\begin{array}{c} 0 \\ \hline B_0 \end{array}\right] u$$

where

$A_0 = \text{diag}\ [1.8,\ 2.7,\ 8.36,\ 8.75,\ 11.5,\ 17.7,\ 21.7,\ 22.6,\ 72.9,\ 85.6,\ 106,\ 167]$

$$B_0 = \begin{bmatrix} -0.023 & -0.067 & -0.439 \\ -0.112 & 0.017 & 0.069 \\ -0.077 & 0.271 & 0.046 \\ 0.189 & -0.050 & -0.249 \\ 0.156 & -0.049 & 0.351 \\ -0.289 & 0.289 & -0.289 \\ -0.320 & -0.369 & -0.049 \\ 0.365 & 0.299 & -0.069 \\ -0.229 & 0.250 & 0.231 \\ 0.167 & -0.150 & -0.317 \\ -0.145 & 0.146 & -0.220 \\ 0.025 & -0.013 & 0.114 \end{bmatrix}$$

The matrix Q_0 to be used in the design procedure outlined in Sec. VI will be taken as a diagonal 12×12 matrix.

In the design iterations presented in Table 4 the diagonal of the matrix Q_0 will be selected. Using the design procedure outlined in Sec. VI, the resulting closed-loop eigenvalue system and its damping ratios can be found.

To illustrate, the symmetric negative-definite output feedback gain matrix for the last iteration of Table 4 is given by

$$K = \begin{bmatrix} -12.426 & -0.506 & 3.129 \\ -0.506 & -7.591 & -5.199 \\ 3.129 & -5.199 & -12.822 \end{bmatrix}$$

VIII. Conclusion

The closed-loop performance properties of a large flexible space structure have been considered. It was shown that when actuators and sensors are colocated, a negative-definite symmetric-output feedback gain matrix will result in a stable closed-loop system. Further, it was shown that these stability properties are not affected by truncated modes and by parameter variations. A design procedure was presented with a numerical example. This design procedure seeks, in a systematic way, the gain matrix to be used in the controller. The search itself is among a set of robust gain matrices which is the underlying property of this design procedure. In addition, treating the colocated controller from the modern control theory point of view allows better understanding of the problem. In particular, one can see what class of optimization problem is achievable with this class of controllers, which goes beyond the question of stability alone.

Appendix: Proof

Sufficiency:

$$P = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha B_0 (B_0^T B_0)^{-1} B_0^T \end{array}\right] + \left[\begin{array}{c|c} \alpha A_0 & 0 \\ \hline 0 & \alpha N_0 N_0^T \end{array}\right]$$

since, from Eqs. (26) and (27)

$$\left[\begin{array}{c} B_0^T \\ \hline N_0^T \end{array}\right]^{-1} = [B_0 (B_0^T B_0)^{-1} \mid N_0]$$

we get

$$P = \left[\begin{array}{c|c} \alpha A_0 & 0 \\ \hline 0 & \alpha I_n \end{array}\right]$$

and, from Eq. (24)

$$Q = \alpha^2 B R^{-1} B^T - PA - A^T P = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha^2 B_0 R^{-1} B_0^T \end{array}\right]$$

Necessity: From Eq. (24)

$$Q = \alpha^2 B R^{-1} B - PA - A^T P$$

Since

$$P = \left[\begin{array}{c|c} E_3 & E_2^T N_0^T \\ \hline N_0 E_2 & B_0 (B_0^T B_0)^{-1} B_0^T + N_0 E_1 N_0 \end{array}\right]$$

we find

$$Q=\left[\begin{array}{c|c} E_2^T N_0 A_0 + A_0 N_0^T E_2 & A_0 [B_0^T (B_0 B)^{-1} B_0^T + N_0 E_1 N_0^T] - E_3 \\ \hline [\alpha B_0 (B_0^T B_0)^{-1} B_0^T + N_0 E_1 N_0^T] A_0 - E_3 & \alpha^2 B_0 R^{-1} B_0^T - N_0 E_2 - E_2 N_0^T \end{array}\right]$$

Since Q is block diagonal, then, using Eqs. (46) and (17), we find

$$E_1 = \alpha I_{n-m} \qquad E_2 = 0 \qquad E_3 = \alpha A_0$$

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